

SPANS OF PREFERENCE FUNCTIONS FOR DE BRUIJN SEQUENCES

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ABSTRACT. A nonbinary Ford sequence is a de Bruijn sequence generated by simple rules that determine the priorities of what symbols are to be tried first, given an initial word of size n which is the order of the sequence being generated. This set of rules is generalized by the concept of a preference function of span $n - 1$, which gives the priorities of what symbols to appear after a substring of size $n - 1$ is encountered. In this paper we characterize preference functions that generate full de Bruijn sequences. More significantly, We establish that any preference function that generates a de Bruijn sequence of order n also generates de Bruijn sequences of all orders higher than n , thus making the Ford sequence no special case. Consequently, we define the preference function complexity of a de Bruijn sequence to be the least possible span of a preference function that generates this de Bruijn sequence.

1. INTRODUCTION

Given a positive integer $t > 1$ and an alphabet $A = \{0, 1, \dots, t - 1\}$ of size t , a de Bruijn sequence of order n over the alphabet A is a sequence of symbols such that every pattern of size n appears exactly once as a block of contiguous symbols. For example, 00110 and 0011221020 are two de Bruijn sequences of order 2 over the alphabets $\{0, 1\}$ and $\{0, 1, 2\}$ respectively. The existence of these sequences for any finite size alphabet and any order is a well known fact [3].

For the binary alphabet, a classical but rather curious algorithm that generates a de Bruijn sequence for any order n is called the “prefer-one” algorithm. It consists of the following simple steps. Begin by writing n zeros. Then for $k > n$, write a one for the k^{th} bit of the sequence if the newly formed n -tuple has not previously appeared in the sequence, otherwise write a zero. This is repeated, preferring one every step of the way, until neither appending one nor zero puts a new n -tuple, at which time the algorithm halts.

The prefer-one sequence is traced back to Martin [9]. But it has been rediscovered by many authors, see Fredricksen [5] for an exposition.

The prefer-one algorithm is generalized to an alphabet of size $t > 2$ by preferring a higher value over a lower value. That is, once the initial n zeros are written, the value $t - 1$ is appended if the word formed by the n most recent symbols is new, otherwise $t - 2$ is proposed, otherwise $t - 3$, etc. This sequence was proposed by Ford [4] and it

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$0 \rightarrow 2, 1, 0$	$0 \rightarrow 1, 2, 0$	$0 \rightarrow 1, 2, 0$
$1 \rightarrow 2, 1, 0$	$1 \rightarrow 1, 0, 2$	$1 \rightarrow 2, 1, 0$
$2 \rightarrow 2, 1, 0$	$2 \rightarrow 2, 1, 0$	$2 \rightarrow 0, 2, 1$
$0 \rightarrow 3, 2, 1, 0$	$0 \rightarrow 1, 2, 3, 0$	$0 \rightarrow 3, 2, 1, 0$
$1 \rightarrow 3, 2, 1, 0$	$1 \rightarrow 3, 1, 0, 2$	$1 \rightarrow 3, 2, 1, 0$
$2 \rightarrow 3, 2, 1, 0$	$2 \rightarrow 0, 2, 1, 3$	$2 \rightarrow 0, 2, 3, 1$
$3 \rightarrow 3, 2, 1, 0$	$3 \rightarrow 2, 3, 1, 0$	$3 \rightarrow 0, 2, 3, 1$

TABLE 1. Some preference diagrams of de Bruijn sequences with alphabet sizes 3 and 4.

therefore bears his name. We will refer to this generalization as the “prefer-higher” algorithm.

In this paper, we show that preferring higher values is not necessary to obtain full de Bruijn sequences. In fact, a binary algorithm similar to the prefer-one was recently proposed in [1]. This algorithm is called the prefer opposite as it proposes a bit that is opposite to the bit most recently appended to the sequence. Although the prefer opposite sequence is not a de Bruijn sequence, it only misses the constant word 1^n . In the non-binary case, other preferences can be constructed that yield full sequences. For example, each diagram in Table 1 can generate a full de Bruijn sequence of arbitrary order n that starts with the initial word 0^n . Each row in a diagram displays the digits to be proposed, in decreasing priority, when the rightmost digit of the sequence being constructed is the digit that appears on the left side of the arrow of that row. A proposed digit is accepted if the most recently formed word of size n has not appeared earlier in the sequence, otherwise the next digit in that row is proposed.

It is worth noticing here that the upper and lower left diagrams give the same decreasing preference regardless of the previous digit. Thus they display the prefer-higher rules for alphabet sizes 3 and 4 respectively.

The following are respectively all the sequences of order 2 that are generated using the diagrams in Table 1.

00221201100; 00110221200; 00120221100;
003323130221201100; 001320221103312300; 003020132233121100;

In the sequel it will be proven that these diagrams generate de Bruijn sequences of all orders. More generally, we will characterize all such diagrams that produce full sequences.

2. MAIN RESULTS

The idea of generating a de Bruijn sequence by making preferences is formalized in the concept of preference functions, defined in Golomb [6] who attributes it to Welsh. In any de Bruijn sequence of order n , a word of size $(n-1)$ appears exactly t times. A preference function gives the priority list of what digits is to come first, second, third, etc. after a word of size $n-1$ appears in the sequence. Here is a precise definition.

Definition 2.1. A preference function P of span $n-1$ is a t -dimensional vector valued function of $n-1$ variables such that, for each choice of the vector $\mathbf{a} = (a_1, \dots, a_{n-1})$ from the set A^{n-1} , the entries of the vector $(P_1(\mathbf{a}), \dots, P_t(\mathbf{a}))$ form a permutation of the elements of A .

Definition 2.2. Given a preference function P , the least preference function induced by P is a function g from A^{n-1} to A^{n-1} defined as

$$g(a_1, \dots, a_{n-1}) = (a_2, \dots, a_{n-1}, P_t(a_1, \dots, a_{n-1})).$$

The following process is given in Golomb [6] and it shows how a preference function of span $s - 1$ is used to construct recursive periodic sequences of order s .

Definition 2.3. For any word (I_1, \dots, I_n) and preference function P of span $n - 1$, the following inductive definition determines a unique finite sequence $\{a_i\}$:

1. $a_1 = I_1, \dots, a_n = I_n$.
2. If $a_{N+1}, \dots, a_{N+n-1}$ have been defined, then $a_{N+n} = P_i(a_{N+1}, \dots, a_{N+n-1})$, where i is the smallest integer such that the word

$$(a_{N+1}, \dots, a_{N+n-1}, P_i(a_{N+1}, \dots, a_{N+n-1}))$$

has not previously appeared as a segment of the sequence (provided that there is such i).

3. Let $L = L\{a_i\}$ be the first value of N such that no i can be found to satisfy item 2. Then a_{L+n-1} is the last digit of the sequence and L is called the cycle period.

Conversely, we remark that any periodic sequence induces at least one preference function whose corresponding sequence is the periodic sequence itself. To see this, consider a periodic sequence S started at the word J_1, \dots, J_n , where n is the smallest word size such that every pattern of size n occurs at most once in S . Now consider all occurrences (if any) of a pattern \mathbf{w} of size $n - 1$ in a single period of S . Since every pattern of size n occurs at most once, the number of occurrences $r(\mathbf{w})$ of the pattern \mathbf{w} is bounded above by t . For $i = 1$ to $r(\mathbf{w})$ let $P_i(\mathbf{w})$ be the digit that occurs right after the i^{th} occurrence of \mathbf{w} . If $r(\mathbf{w}) < t$ let $P_{r(\mathbf{w})+1}(\mathbf{w}), \dots, P_t(\mathbf{w})$ be any permutation of the digits which do not appear as entries of $(P_1(\mathbf{w}), \dots, P_r(\mathbf{w}))$.

By the above construction, it is evident that the preference function P along with the initial word J_1, \dots, J_n produces the sequence S . The next proposition follows immediately by the above discussion.

Proposition 2.4. *Fixing an initial word (I_1, \dots, I_n) , there is a one to one correspondence between the set of de Bruijn sequences of order n and the set of preference functions of span $n - 1$ which generate de Bruijn sequences of order n started at (I_1, \dots, I_n) .*

Given an arbitrary preference function, a natural question is whether or not this preference function generates a full de Bruijn sequence. In this section, we take on the problem of characterizing such *complete* preference functions.

For completeness, we now state two theorems, given in Golomb [6], which present conditions on a preference function to produce a de Bruijn sequence.

Definition 2.5. For $0 \leq r \leq n - 1$, we say that (x_1, \dots, x_{n-1}) has an r -overlap with (I_1, \dots, I_n) if $(x_{n-r}, \dots, x_{n-1}) = (I_1, \dots, I_r)$. Notice that any $(n - 1)$ -digit word at least has a zero overlap with (I_1, \dots, I_n) .

Theorem 16. (of Golomb's Chapter VI) For any initial word (I_1, \dots, I_n) , if P is a preference function of span $n - 1$ that satisfies $P_t(x_1, \dots, x_{n-1}) = I_{r+1}$, when r is the

largest integer such that (x_1, \dots, x_{n-1}) has an r -overlap with (I_1, \dots, I_n) , then the sequence generated by (I_1, \dots, I_n) and P has length t^n , i.e., it is a de Bruijn sequence of order n .

While the previous theorem states a condition that guarantees that a preference function of span $n - 1$ produces a de Bruijn sequence of order n , the next theorem starts with a preference function that is known to generate a de Bruijn sequence of order $n - 1$ and provides a way to construct a preference function of span $n - 1$ that produces a de Bruijn sequence of order n . This *recursive construction* is stated and proved for the binary case in Golomb [6], although it is claimed that the theorem can be easily generalized to the non-binary case.

Theorem 17. (of Golomb's Chapter VI) The following hypotheses are adopted:

1. Let (I_1, \dots, I_n) be an arbitrary initial word.
2. Let $P(x_1, \dots, x_{n-2}) = (P_1, P_2)$ be the preference function for the binary de Bruijn sequence of order $n-1$, $\{b_i\}$, and initial word (I_1, \dots, I_{n-1}) , such that $P_1(I_1, \dots, I_{n-2}) = 1 + I_{n-1} \pmod 2$.
3. Let $x_1 \oplus F(x_2, \dots, x_{n-1})$ be the feedback formula for $\{b_i\}$. That is, $b_i = b_{i-n+1} \oplus F(x_{i-n+2}, \dots, x_{i-1})$ for all i .
4. Let $P^*(x_1, \dots, x_{n-1}) = (P_1^*, P_2^*)$ which satisfies $P_2^* = 1 \oplus P_1^*$ and $0 = [P_1^*(x_1, \dots, x_{n-1}) \oplus 1 \oplus x_1 \oplus F(x_2, \dots, x_{n-1})] \times [x_1 \oplus F(x_2, \dots, x_{n-1}) \oplus P_1(x_2, \dots, x_{n-1})]$.

where the \oplus is taken as addition modulo 2. It follows from these hypotheses that the sequence $\{a_i\}$, generated by (I_1, \dots, I_n) and P^* , is a de Bruijn sequence of order n .

While the conditions stated in Theorem 17 indeed generate a complete binary preference table, these conditions are in a sense artificially designed to make possible the inductive proof given in Golomb [6]. In what follows we will show that, for any alphabet size t , a preference function of span $n - 1$ is itself capable of generating de Bruijn sequences of all orders larger than or equal to n . Before we do this we will characterize preference functions of span $n - 1$ that generate de Bruijn sequences of order n , i.e., complete preference functions.

A de Bruijn sequence of order n can be started with any of its words of size n . Unless otherwise stated, in the rest of this paper we will only be concerned with an initial word $(I_1, \dots, I_n) = 0^n$, i.e. the constant string of n zeros.

Definition 2.6. Let E be a finite set, let f be a function from E to itself and let $l \geq 1$ be an integer. By a cycle of length l induced by f we mean a sequence of elements x_1, \dots, x_l such that $f(x_i) = x_{i+1}$ for $i = 1$ to $l - 1$ and $f(x_l) = x_1$.

We now state our first main result.

Theorem 2.7. *Let P be a complete preference function of span $n - 1$ that corresponds to a de Bruijn sequence started at the string 0^n . Then the least preference function $g(x_1, \dots, x_{n-1})$ has no cycles of any length except the self-loop $(0^{n-1}, 0^{n-1})$, i.e. $g(0^{n-1}) = 0^{n-1}$, which must be a cycle of g .*

Proof. Let S be the de Bruijn sequence starting with 0^n and resulting from P . If $(0^{n-1}, 0^{n-1})$ is not a cycle of g then $P_t(0^{n-1}) = a \neq 0$ and hence 0 has a higher preference over a , that is $P_i(0^{n-1}) = 0$ for some $i < t$. Since S is a de Bruijn sequence of order n , the word $0^{n-1}a$ must be a substring. This means that the word

$(0^{n-1}, P_i(0^{n-1}) = 0^n$ must have been proposed and accepted earlier in the sequence. So that 0^n occurs twice in the sequence, which is a contradiction. Suppose now that g has a cycle of length i , $1 \leq i \leq t^{n-1} - 1$ other than the self loop at 0^{n-1} . Namely, suppose that for some y_1, \dots, y_{i+n-1}

$$(2.1) \quad \begin{aligned} g(y_1, \dots, y_{n-1}) &= (y_2, \dots, y_n) \\ g(y_2, \dots, y_n) &= (y_3, \dots, y_{n+1}) \\ &\vdots \\ g(y_i, \dots, y_{i+n-2}) &= (y_{i+1}, \dots, y_{i+n-1}). \end{aligned}$$

where $(y_{i+1}, \dots, y_{i+n-1}) = (y_1, \dots, y_{n-1})$ but $(y_j, \dots, y_{j+n-2}) \neq (y_k, \dots, y_{k+n-2})$ for all pairs (j, k) such that $1 \leq j < k \leq i + 1$ and $(j, k) \neq (1, i + 1)$.

Since S is a de Bruijn sequence of order n , (y_1, \dots, y_n) occurs in S . By definition of g and the first equation in Display (2.1), $y_n = P_t(y_1, \dots, y_{n-1})$. It follows that all the words (y_1, \dots, y_{n-1}, z) , $z \neq y_n$ must have occurred earlier in the sequence. This implies that all the predecessors (y, y_1, \dots, y_{n-1}) , for $y \in A$ have occurred before (y_1, \dots, y_n) . In particular $(y_i, \dots, y_{i+n-1}) = (y_i, y_1, \dots, y_{n-1})$ has occurred before (y_1, \dots, y_n) .

Now, $g(y_i, \dots, y_{i+n-2}) = (y_{i+1}, \dots, y_{i+n-1})$ is equivalent to $P_t(y_i, \dots, y_{i+n-2}) = y_{i+n-1}$. Using the above argument, we see that all the words $(y_i, \dots, y_{i+n-2}, z)$, $z \neq y_{i+n-1}$, and therefore all their predecessors $(y, y_i, \dots, y_{i+n-2})$ must have occurred earlier in the sequence. In particular, $(y_{i-1}, y_i, \dots, y_{i+n-2})$ occurs before (y_i, \dots, y_{i+n-1}) , which was just shown to occur before (y_1, \dots, y_n) . Repeating the same reasoning a total of i times, we see that (y_1, \dots, y_n) must have occurred earlier in the sequence S . That is, (y_1, \dots, y_n) occurs more than once in S , contradicting the assumption that the latter is a de Bruijn sequence. This establishes that g has no cycles besides $(0^{n-1}, 0^{n-1})$. \square

It is important to remark here that in the above theorem, the initial word must be 0^n or—of course—a constant string i^n (in which case the self loop of g must be (i^{n-1}, i^{n-1})). Indeed, Table 2 displays a de Bruijn sequence of alphabet size 3 and order 3 with its corresponding preference function. Noting that the initial word is 001, we can see that the induced least preference function g has the cycle $(00, 01, 10, 00)$.

00	→	0,2,1	10	→	1,2,0	20	→	1,0,2
01	→	1,2,0	11	→	0,1,2	21	→	2,1,0
02	→	0,1,2	12	→	1,2,0	22	→	2,0,1

TABLE 2. Preference rules for the sequence 00110121222010200021112022100.

The converse of Theorem 2.7 is also true. That is, if a given preference function of span $n - 1$ induces a least preference function g that has no cycles except the self loop at 0^{n-1} , then the preference function produces a de Bruijn cycle of order n started at the word 0^n . However, the next result is much stronger than this converse. We state it after the following algorithm.

Algorithm P

Input: Two integers $s > 1$ and $n \geq s$ and a preference function P of span $s - 1$.

Output: a unique de Bruijn sequence $S = \{a_i\}$ of order n .

1. $a_1 = 0, \dots, a_n = 0$.
2. If $a_{N+1}, \dots, a_{N+n-1}$ have been defined, then $a_{N+n} = P_i(a_{N+n-s+1}, \dots, a_{N+n-1})$, where i is the smallest integer between 1 and t such that the word

$$(a_{N+1}, \dots, a_{N+n-1}, P_i(a_{N+n-s+1}, \dots, a_{N+n-1}))$$

has not previously appeared as a segment of the sequence (provided that there are such i).

3. Let $L = L\{a_i\}$ be the smallest value of N such that no i can be found to satisfy the condition in (2). Then a_{L+n-1} is the last digit of the sequence and L is called the cycle period.

Theorem 2.8. *Let P be a preference function of span $s - 1$ that induces a least preference function g which admits no cycles except the self loop $(0^{s-1}, 0^{s-1})$. Then for any integer $n \geq s$ the sequence given by Algorithm P is a de Bruijn sequence of order n .*

We observe that this theorem establishes that the Ford sequence is rather the norm than the exception. For the Ford sequence, the permutation $(t - 1, t - 2, \dots, 0)$ —which is a de Bruijn sequence of order 1—generates de Bruijn sequences of all orders. Using Theorem 2.8, given any de Bruijn sequence of order s , we can construct the corresponding preference function of span $s - 1$ which in turn can generate a unique de Bruijn sequence of any order higher than s . The proof of this theorem will be given after a few lemmas are formulated and proved.

Lemma 2.9. *The sequence S in Theorem 2.8 ends just after the word $a0^{n-1}$ is encountered, for some $a \in A$, $a \neq 0$.*

Proof. First, it is immediate by the construction in **Algorithm P** that a word of size n occurs at most once in the constructed sequence. Suppose now that the algorithm terminates just after the word $(x_1, \dots, x_n) \neq a0^{n-1}$ is realized. That is, (x_2, \dots, x_n, y) must have appeared earlier in the sequence for all $y \in A$. This implies that (x_2, \dots, x_n) appeared $t + 1$ times. Since (x_2, \dots, x_n) is not equal to 0^{n-1} , it is not the initial block of the sequence so that every time it appeared it was preceded by something. The pigeon hole principle thus implies that there exists an element $z \in A$ such that (z, x_2, \dots, x_n) occurs twice in the sequence, which is a contradiction. \square

Lemma 2.10. *All words of the form $b0^{n-1}$ occur in the sequence S of Theorem 2.8.*

Proof. By Lemma 2.9 the sequence ends with the word $a0^{n-1}$. The word $0^n = 0^{n-1}0$, which already occurs in the beginning, can not be appended after $a0^{n-1}$. Since $P_t(0^{n-1}) = 0$, no other symbol z can be appended either. This implies that all words of the form $0^{n-1}z$ have occurred earlier in the sequence. It follows that all the words of the form $b0^{n-1}$, where $b \neq 0$ must occur in the sequence. \square

Lemma 2.11. *If $X_1 = (x_1, \dots, x_n)$ is a word that does not occur in S then neither does the word $X_2 = (x_2, \dots, x_n, c)$, where $c = P_t(x_{n-s+2}, \dots, x_n)$.*

Proof. Suppose that X_2 occurs in S . X_2 can not be the zero string 0^n because the latter occurs as the first string. Hence X_2 is preceded by some string. Since $c = P_t(x_{n-s+2}, \dots, x_n)$ has the least preference, it follows that all the words (x_2, \dots, x_n, z) must have occurred earlier in the sequence. Therefore the set of all predecessors

(y, x_2, \dots, x_n) must have occurred for all values of y . In particular, $X_1 = (x_1, \dots, x_n)$ must have occurred, which is a contradiction. \square

Proof. (of Theorem 2.8) Suppose that (x_1, \dots, x_n) is a pattern that does not appear in S .

Case 1. Let us first suppose that $(x_{n-s+2}, \dots, x_n) = 0^{s-1}$. Since (x_1, \dots, x_n) can not be all zeros, there must exist an integer i , $1 \leq i \leq n - s + 1$ such that $x_i \neq 0$ but $x_j = 0$ for all j , $i + 1 \leq j \leq n$. Since $g(0^{s-1}) = 0^{s-1}$ it is clear that $P_t(0^{s-1}) = 0$. It follows by applying Lemma 2.11 that $(x_2, \dots, x_i, 0^{n-i+1})$ does not occur in S . By the same argument, applying Lemma 2.11 another $(i - 2)$ times, we see that the word $x_i, 0^{n-1}$ does not occur. This contradicts Lemma 2.10.

Case 2. Suppose now that $(x_{n-s+2}, \dots, x_n) \neq 0^{s-1}$. Then by Lemma 2.11 the word $(x_2, \dots, x_n, x_{n+1})$ does not appear either, where $x_{n+1} = P_t(x_{n-s+2}, \dots, x_n)$. Moreover, $(x_{n-s+2}, \dots, x_n) \neq (x_{n-s+3}, \dots, x_{n+1})$ for otherwise the least preference function g would have a cycle of length 1 that is distinct from the self loop $(0^{s-1}, 0^{s-1})$, namely

$$(x_{n-s+1}, \dots, x_n, g(x_{n-s+1}, \dots, x_n)) = (x_{n-s+2}, \dots, x_{n+1}),$$

which can not be the case by the given.

If $(x_{n-s+3}, \dots, x_{n+1}) = 0^{s-1}$, Case 1 above leads to a contradiction. So it is safe to assume that this is not the case. We claim that, by applying this argument repeatedly, we eventually get a word ending with 0^{s-1} that does not occur in S . To see this note that, after i repetitions of Lemma 2.11—with $i \leq t^{s-1}$, we conclude that the word $(x_{n+i-s+2}, \dots, x_{n+i})$ does not occur in S , where for $j = n + 1$ to $n + i$, $x_j = P_t(x_{j-s+1}, \dots, x_{j-1})$ and $(x_{j-s+2}, \dots, x_j) \neq 0^{s-1}$. Since $g(x_{j-s+1}, \dots, x_{j-1}) = (x_{j-s+2}, \dots, x_j)$ and since g has no cycles of any length (namely, no cycles of length $1, 2, \dots, t^{s-1} - 1$) other than the self loop at 0^{s-1} , we see that $(x_{j-s+1}, \dots, x_j) \neq (x_{j'-s+1}, \dots, x_{j'})$ for all $j < j'$ and $n \leq j, j' \leq n + i$. Otherwise, the sequence

$$(x_{j-s+1}, \dots, x_j), (x_{j-s+2}, \dots, x_{j+1}), \dots, (x_{j'-s+1}, \dots, x_{j'})$$

would form a cycle of length $j' - j$.

For any i such that $1 \leq i \leq t^{s-1} - 2$, if the right tail $(x_{n+i-s+2}, \dots, x_{n+i}) = 0^{s-1}$ then applying Case 1 leads to a contradiction. Suppose then that the right tail is distinct from 0^{s-1} for all $i = 1$ to $t^{s-1} - 2$. Then, for $i = t^{s-1} - 1$, the facts that all the words are distinct and that there are $i + 1 = t^{s-1}$ words imply that the last word of size $s - 1$ is necessarily equal to 0^{s-1} , thus leading to a contradiction, by Case 1. This establishes the theorem. \square

3. PREFERENCE FUNCTION COMPLEXITY

In the vast literature on de Bruijn sequences, there has been more than one method to classify these sequences. One well known criterion for binary de Bruijn sequences is the number of ones in the truth table of the corresponding feedback function, (namely, the function F defined in the statement of item (3) in Theorem 17 above).

Also, de Bruijn sequences have been classified according to their linear complexity, which is defined as the minimal span of a linear shift register that generates the de Bruijn sequence. In other words, it is the minimal integer N such that there exists a linear feedback function $F = F(x_2, \dots, x_N)$ that can generate the de Bruijn sequence.

It was proven by Chan, Games and Key [2] that the linear complexity of a binary de Bruijn sequence of order n is between $2^{n-1} + n$ and $2^n - 1$.

In this section, we use Theorem 2.8 to introduce a new notion of complexity of de Bruijn sequences of any alphabet size that relates to the preference function which generates the sequence. We thus obtain another classification of de Bruijn sequences based on this complexity.

To fix ideas, we observe that, by Theorem 2.8, it is clear that an algorithm such as the one that generates the Ford sequence (for general alphabet size t) is rather the norm than the exception. For the latter algorithm, a preference function that generates a de Bruijn sequence of order 1 also generates a de Bruijn sequence of any order n larger than 1 when started with the initial word 0^n .

Let us also observe that the preference function of the Ford sequence—with the “prefer-higher” algorithm—is a constant function of span 0 that is given by $P(x) = (t - 1, t - 2, \dots, 0)$ for all x in A . See Table 1. The corresponding least preference function is given by $g(x) \equiv 0$, which admits the only cycle $g(0) = 0$. In fact, the two preference diagrams in the leftmost column of Table 1 have span zero while the remaining diagrams have span one.

Definition 3.1. Given an order n de Bruijn sequence S that starts with the fixed word 0^n we define the preference function complexity $comp_0(S)$ as the smallest integer s , $0 \leq s \leq n$ such that there exists a preference function of span s that generates the sequence S with the initial word 0^n .

The sixteen binary de Bruijn sequences of order four are given in Table 3 while their corresponding preference functions are given in Table 4. Notice that sequence (3)—the Ford sequence—does not depend on any of the previous three bits so it has preference function span 0 while sequence (6) depends only on the previous two bits so it has span 2. All other sequences have preference function span 3. Thus they have full span. There are no sequences with span 1, due to the binary alphabet. Sequence (2) comes close. In fact, changing the preference of ‘111’ to 1 then 0 makes the preference function depend only on the previous bit but this introduces a self loop $111 \rightarrow 111$ in the corresponding least preference function so the resulting sequence misses the word 1111. Note that this sequence is the prefer opposite sequence mentioned earlier.

Proposition 3.2. *The distribution of de Bruijn sequences of order n according to their preference function complexity is given by $N_0(n) = (t - 1)!$, $N_1(n) = ((t - 1)!)^t \cdot t^{t-2}$, and for $i > 1$ $N_i(n) = ((t - 1)!)^{t^i} \cdot t^{t^i - i - 1} - ((t - 1)!)^{t^{i-1}} \cdot t^{t^{i-1} - i}$ where, for $i = 0$ to $n - 1$,*

$$N_i(n) = \text{card}\{S : S \text{ is a de Bruijn sequence of order } n \text{ such that } comp(S) = i\}.$$

Proof. For $i = 0$ the order of preference does not depend on any of the previous digits, in particular it does not depend on the immediately previous digit. Since the only allowed cycle in the induced least preference function g of is the self loop from 0 to 0 it follows that $g(i) = 0$ for all digits i . The remaining $t - 1$ digits can be given any of $(t - 1)!$ orders of preference.

For $i \geq 1$, it is evident that the complexity of a de Bruijn sequence of order i does not exceed $i - 1$. Moreover, it is well known, see [8], that the total number of de Bruijn sequences of order i is given by the formula $M(t, i) = [(t - 1)!]^{t^{i-1}} \cdot t^{t^{i-1} - i}$. Hence $N_1(1)$ is $M(t, 1)$ minus the number of sequences of complexity 0.

1	0000100110101111000	9	0000101111001101000
2	0000101001101111000	10	0000101111010011000
3	0000111101100101000	11	0000101100111101000
4	0000111101011001000	12	0000110010111101000
5	0000100111101011000	13	0000111101001011000
6	0000101001111011000	14	0000110100101111000
7	0000110111100101000	15	0000101101001111000
8	0000110101111001000	16	0000111100101101000

TABLE 3. Binary de Bruijn sequences of order 4.

Similarly, $N_i(i)$ is $M(t, i)$ minus the number of sequences whose complexity is less than i . Since Theorem 2.8 implies that every preference function of span $i < n$ also produces a de Bruijn sequence of order n , it follows that $N_i(n) = N_i(i)$. \square

We will say that two sequences $\{a_i\}$ and $\{b_i\}$ are equivalent if $b_i = \sigma(a_i)$ for some permutation σ of the alphabet A . Our last result relates to de Bruijn sequences with complexity zero. Notice that while the binary case allows only one preference function with zero span, higher values of t yield $(t-1)!$ cases. The following proposition shows that in fact all of these cases yield equivalent de Bruijn sequences.

Proposition 3.3. *All de Bruijn sequences of preference function complexity zero are equivalent, up to a permutation of the digits, to the Ford sequence.*

Proof. Let Q be an arbitrary complete preference function of span zero. Evidently, there exists a permutation σ such that $Q_i = (\sigma(t-1), \dots, \sigma(0))$ for all $i \in A$. Let $\{b_i\}$ be the sequence of order n —started at 0^n —that corresponds to Q . Consider now the sequence $\{a_i\}$ defined by $a_i = \sigma^{-1}(b_i)$, which is obviously a de Bruijn sequence. We claim that $\{a_i\}$ is the Ford sequence of order n . To see this, let $i_1 < i_2 < \dots < i_t$ be the positions of a pattern x_1, \dots, x_{n-2} in the sequence $\{a_i\}$. It follows that i_1, \dots, i_t are the positions of the pattern $\sigma(x_1), \dots, \sigma(x_{n-2})$ in the sequence $\{b_i\}$. By definition of Q , the substrings

$$(b_{i_1}, \dots, b_{i_1+n-2}), \dots, (b_{i_t}, \dots, b_{i_t+n-2})$$

are followed respectively by $\sigma(t-1), \dots, \sigma(0)$. Therefore, the substrings

$$(a_{i_1}, \dots, a_{i_1+n-2}), \dots, (a_{i_t}, \dots, a_{i_t+n-2})$$

of $\{a_i\}$ are followed by $t-1, t-2, \dots, 0$.

Since this is true for any pattern (x_1, \dots, x_{n-1}) , the proof is complete. \square

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	sequence															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
000	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0
001	0,1	0,1	1,0	1,0	0,1	0,1	1,0	1,0	0,1	0,1	0,1	1,0	1,0	1,0	0,1	1,0
010	0,1	1,0	1,0	1,0	0,1	1,0	1,0	1,0	1,0	1,0	1,0	1,0	0,1	0,1	1,0	1,0
011	0,1	0,1	1,0	1,0	1,0	1,0	0,1	1,0	1,0	1,0	0,1	0,1	1,0	0,1	0,1	1,0
100	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0
101	0,1	0,1	1,0	0,1	0,1	0,1	1,0	0,1	1,0	1,0	1,0	1,0	0,1	0,1	1,0	1,0
110	1,0	1,0	1,0	1,0	1,0	1,0	0,1	1,0	0,1	1,0	0,1	0,1	1,0	1,0	1,0	0,1
111	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0
span	3	3	0	3	3	2	3	3	3	3	3	3	3	3	3	3

TABLE 4. preference functions of order 4 binary de Bruijn sequences and their spans.

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